

Chapter 1

Supersymmetry in Random Matrix Theory

THOMAS GUHR¹

¹Fakultät für Physik, Universität Duisburg–Essen,
Lotharstrasse 1, 47057 Duisburg, Germany

Abstract

Supersymmetry is nowadays indispensable for many problems in Random Matrix Theory. It is presented here with an emphasis on conceptual and structural issues. An introduction to supermathematics is given. The Hubbard–Stratonovich transformation as well as its generalization and superbosonization are explained. The supersymmetric non-linear σ model, Brownian motion in superspace and the color–flavor transformation are discussed.

1.1 Generating Functions

We consider $N \times N$ matrices H in the three symmetry classes [Dys62] real symmetric, Hermitean or quaternion real, that is, self-dual Hermitean. The Dyson index β takes the values $\beta = 1, 2, 4$, respectively. For $\beta = 4$, the $N \times N$ matrix H has 2×2 quaternion entries and all its eigenvalues are doubly degenerate. For a given symmetry, an ensemble of random matrices is specified by choosing a probability density function $P(H)$ of the matrix H . The ensemble is referred to as invariant or rotation invariant if

$$P(V^{-1}HV) = P(H) \tag{1.1.1}$$

where V is a fixed element in the group diagonalizing H , that is, in $\text{SO}(N)$, $\text{SU}(N)$ or $\text{USp}(2N)$ for $\beta = 1, 2, 4$, respectively. Equation (1.1.1) implies that the probability density function only depends on the eigenvalues,

$$P(H) = P(X) = P(x_1, \dots, x_N) . \quad (1.1.2)$$

Here, we write the diagonalization of the random matrix as $H = U^{-1}XU$ with $X = \text{diag}(x_1, \dots, x_N)$ for $\beta = 1, 2$ and $X = \text{diag}(x_1, x_1, \dots, x_N, x_N)$ for $\beta = 4$. The k -point correlation function $R_k(x_1, \dots, x_k)$ measures the probability density of finding a level around each of the positions x_1, \dots, x_k , the remaining levels not being observed. One has [Dys62, Meh04]

$$R_k(x_1, \dots, x_k) = \frac{N!}{(N-k)!} \int_{-\infty}^{+\infty} dx_{k+1} \cdots \int_{-\infty}^{+\infty} dx_N |\Delta_N(X)|^\beta P(X) , \quad (1.1.3)$$

where $\Delta_N(X)$ is the Vandermonde determinant. If the probability density function factorizes,

$$P(X) = \prod_{n=1}^N P^{(E)}(x_n) , \quad (1.1.4)$$

with a probability density function $P^{(E)}(x_n)$ for each of the eigenvalues, the correlation functions (1.1.3) can be evaluated with the Mehta–Mahoux theorem [Meh04]. They are $k \times k$ determinants for $\beta = 2$ and $2k \times 2k$ quaternion determinants for $\beta = 1, 4$ whose entries, the kernels, depend on only two of the eigenvalues x_1, \dots, x_k .

Formula (1.1.3) cannot serve as the starting point for the Supersymmetry method. A reformulation employing determinants is called for, because these can be expressed as Gaussian integrals over commuting or anticommuting variables, respectively. The key object is the resolvent, that is, the matrix $(x_p^- - H)^{-1}$ where the argument is given a small imaginary increment, $x_p^- = x_p - i\varepsilon$. The k -point correlation functions are then defined as the ensemble averaged imaginary parts of the traces of the resolvents at arguments x_1, \dots, x_k ,

$$R_k(x_1, \dots, x_k) = \frac{1}{\pi^k} \int P(H) \prod_{p=1}^k \text{Im tr} \frac{1}{x_p^- - H} d[H] . \quad (1.1.5)$$

The necessary limit $\varepsilon \rightarrow 0$ is suppressed throughout in our notation. We write $d[\cdot]$ for the volume element of the quantity in square brackets, that is, for the product of the differentials of all independent variables. The definitions (1.1.3) and (1.1.5) are equivalent, but not fully identical. Formula (1.1.5) yields a sum of terms, only one coincides with the definition (1.1.3), all others contain at least one δ function of the form $\delta(x_p - x_q)$, see Ref. [Guh98].

Better suited for the Supersymmetry method than the correlation functions (1.1.5) are the correlation functions

$$\hat{R}_k(x_1, \dots, x_k) = \frac{1}{\pi^k} \int d[H] P(H) \prod_{p=1}^k \text{tr} \frac{1}{x_p - iL_p \varepsilon - H} \quad (1.1.6)$$

which also contain the real parts of the resolvents. The correlation functions (1.1.5) can always be reconstructed, but the way how this is conveniently done differs for different variants of the Supersymmetry method. In Eq. (1.1.5), all imaginary increments are on the same side of the real axis. In Eq. (1.1.6), however, we introduced quantities L_p which determine the side of the real axis where the imaginary increment is placed. They are either $+1$ or -1 and define a metric L . Hence, depending on L , there is an overall sign in Eq. (1.1.6) which we suppress. We use the short hand notations $x_p^\pm = x_p - iL_p \varepsilon$ in the sequel. In some variants of the Supersymmetry method, it is not important where the imaginary increments are, in the supersymmetric non-linear σ model, however, it is of crucial importance. We return to this point.

To prepare the application of Supersymmetry, one expresses the correlation functions (1.1.6) as derivatives

$$\hat{R}_k(x_1, \dots, x_k) = \frac{1}{(2\pi)^k} \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k(x + J) \Big|_{J_p=0} \quad (1.1.7)$$

of the generating function

$$Z_k(x + J) = \int d[H] P(H) \prod_{p=1}^k \left(\frac{\det(H - x_p + iL_p \varepsilon - J_p)}{\det(H - x_p + iL_p \varepsilon + J_p)} \right)^\gamma \quad (1.1.8)$$

with respect to source variables J_p , $p = 1, \dots, k$. For $\beta = 1, 2$ one has $\gamma = 1$ whereas $\gamma = 2$ for $\beta = 4$. For later purposes, we introduce the $2k \times 2k$ matrices $x = \text{diag}(x_1, x_1, \dots, x_k, x_k)$ and $J = \text{diag}(-J_1, +J_1, \dots, -J_k, +J_k)$ for $\beta = 2$ as well as the $4k \times 4k$ matrices $x = \text{diag}(x_1, x_1, x_1, x_1, \dots, x_k, x_k, x_k, x_k)$ and $J = \text{diag}(-J_1, -J_1, +J_1, +J_1, \dots, -J_k, -J_k, +J_k, +J_k)$ for $\beta = 1, 4$, which appear in the argument of Z_k . We write $x^\pm = x - iL\varepsilon$. Importantly, the generating function is normalized at $J = 0$, that is, $Z_k(x) = 1$.

1.2 Supermathematics

Martin [Mar59] seems to have written the first paper on anticommuting variables in 1959. Two years later, Berezin introduced integrals over anticommuting variables when studying second quantization. His posthumously published book [Ber87] is still the standard reference on supermathematics.

1.2.1 Anticommuting Variables

We introduce Grassmann or anticommuting variables ζ_p , $p = 1, \dots, k$ by requiring the relation

$$\zeta_p \zeta_q = -\zeta_q \zeta_p, \quad p, q = 1, \dots, k. \quad (1.2.1)$$

In particular, this implies $\zeta_p^2 = 0$. These variables are purely formal objects. In contrast to commuting variables, they do not have a representation as numbers. The inverse of an anticommuting variable cannot be introduced in a meaningful way. Commuting and anticommuting variables commute. The product of an even number of anticommuting variables is commuting,

$$(\zeta_p \zeta_q) \zeta_r = \zeta_p \zeta_q \zeta_r = -\zeta_p \zeta_r \zeta_q = +\zeta_r \zeta_p \zeta_q = \zeta_r (\zeta_p \zeta_q). \quad (1.2.2)$$

We view the anticommuting variables as complex and define a complex conjugation, ζ_p^* is the complex conjugate of ζ_p . The variables ζ_p and ζ_p^* are independent in the same sense in which an ordinary complex variable and its conjugate are independent. The property (1.2.1) also holds for the complex conjugates as well as for mixtures, $\zeta_p \zeta_q^* = -\zeta_q^* \zeta_p$. There are two different but equivalent ways to interpret $(\zeta_p^*)^*$. The usual choice in physics is

$$(\zeta_p^*)^* = \zeta_p^{**} = -\zeta_p, \quad p = 1, \dots, k, \quad (1.2.3)$$

which has to be supplemented by the rule

$$(\zeta_p \zeta_q \cdots \zeta_r)^* = \zeta_p^* \zeta_q^* \cdots \zeta_r^*. \quad (1.2.4)$$

There is a concept of reality, since we have

$$(\zeta_p^* \zeta_p)^* = \zeta_p^{**} \zeta_p^* = -\zeta_p \zeta_p^* = \zeta_p^* \zeta_p. \quad (1.2.5)$$

Hence, we may interpret $\zeta_p^* \zeta_p$ as the modulus squared of the complex anticommuting variable ζ_p . Alternatively, one can use the plus sign in Eq. (1.2.3) and reverse the order of the anticommuting variables on the right hand side of Eq. (1.2.4). In particular, this also preserves the property (1.2.5).

Because of $\zeta_p^2 = 0$ and since inverse anticommuting variables do not exist, functions of anticommuting variables can only be finite polynomials,

$$f(\zeta_1, \dots, \zeta_k, \zeta_1^*, \dots, \zeta_k^*) = \sum_{\substack{m_p=0,1 \\ l_p=0,1}} f_{m_1 \dots m_k l_1 \dots l_k} \zeta_1^{m_1} \cdots \zeta_k^{m_k} (\zeta_1^*)^{l_1} \cdots (\zeta_k^*)^{l_k} \quad (1.2.6)$$

with commuting coefficients $f_{m_1 \dots m_k l_1 \dots l_k}$. Thus, just like functions of matrices, functions of anticommuting variables are power series. For example, we have

$$\exp(a \zeta_p^* \zeta_p) = 1 + a \zeta_p^* \zeta_p = \frac{1}{1 - a \zeta_p^* \zeta_p}. \quad (1.2.7)$$

where a is a commuting variable.

1.2.2 Vectors and Matrices

A supermatrix σ is defined via block construction,

$$\sigma = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix}, \quad (1.2.8)$$

where a and b are matrices with ordinary complex commuting entries while the matrices μ and ν have complex anticommuting entries. Apart from the restriction that the blocks must match, all dimensions of the matrices are possible. Of particular interest are quadratic $k_1/k_2 \times k_1/k_2$ supermatrices, that is, a and b have dimensions $k_1 \times k_1$ and $k_2 \times k_2$, respectively, μ and ν have dimensions $k_1 \times k_2$ and $k_2 \times k_1$. A quadratic supermatrix σ can have an inverse σ^{-1} . Equally important are supervectors, which are defined as special supermatrices consisting of only one column. As seen in Eq. (1.2.8), there are two possibilities

$$\psi = \begin{bmatrix} z \\ \zeta \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \zeta \\ z \end{bmatrix}, \quad (1.2.9)$$

where z is a k_1 component vector of ordinary complex commuting entries z_p , and ζ is a k_2 component vector of complex anticommuting entries ζ_p . In the sequel we work with the first possibility, but everything to be said is valid for the second one accordingly. The standard rules of matrix addition and multiplication apply, if everything in Sec. 1.2.1 is taken into account. Consider for example the supervector ψ' given by

$$\psi' = \sigma\psi = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} az + \mu\zeta \\ \nu z + b\zeta \end{bmatrix}, \quad (1.2.10)$$

which has the same form as ψ . Hence the linear map (1.2.10) transforms commuting into anticommuting degrees of freedom and vice versa.

The transpose σ^T and the Hermitean conjugate σ^\dagger are defined as

$$\sigma^T = \begin{bmatrix} a^T & -\nu^T \\ \mu^T & b^T \end{bmatrix} \quad \text{and} \quad \sigma^\dagger = (\sigma^T)^*. \quad (1.2.11)$$

The minus sign in front of ν^T ensures that $(\sigma_1\sigma_2)^T = \sigma_2^T\sigma_1^T$ carries over to supermatrices σ_1 and σ_2 . Importantly, $(\sigma^\dagger)^\dagger = \sigma$ always holds, but $(\sigma^T)^T$ is in general not equal to σ . As a special application, we define the scalar product $\psi^\dagger\chi$ where each of the supervectors ψ and χ has either the first or the second of the forms (1.2.9). Because of the reality property (1.2.3), the scalar product $\psi^\dagger\psi$ is real and can be viewed as the length squared of the supervector ψ .

To have cyclic invariance, the supertrace is defined as

$$\text{str } \sigma = \text{tr } a - \text{tr } b \quad (1.2.12)$$

such that $\text{str } \sigma_1 \sigma_2 = \text{str } \sigma_2 \sigma_1$ for two different supermatrices σ_1 and σ_2 . Correspondingly, the superdeterminant is multiplicative owing to the definition

$$\text{sdet } \sigma = \frac{\det(a - \mu b^{-1} \nu)}{\det b} = \frac{\det a}{\det(b - \nu a^{-1} \mu)} \quad (1.2.13)$$

for $\det b \neq 0$ such that $\text{sdet } \sigma_1 \sigma_2 = \text{sdet } \sigma_1 \text{sdet } \sigma_2$.

1.2.3 Groups and Symmetric Spaces

For an introduction to this topic see chapter 3. Here we only present the salient features in the context of the supersymmetry method. The theory of Lie superalgebras was pioneered by Kac [Kac77]. Although the notion of supergroups, particularly Lie supergroups, seems to be debated in mathematics, a consistent definition from a physics viewpoint is possible and — as will become clear later on — urgently called for. All supermatrices u which leave the length of the supervector ψ invariant form the unitary supergroup $U(k_1/k_2)$. With $\psi' = u\psi$ we require $(\psi')^\dagger \psi' = \psi^\dagger u^\dagger u \psi = \psi^\dagger \psi$ and the corresponding equation for $\psi' = u^\dagger \psi$. Hence we conclude

$$u^\dagger u = 1, \quad uu^\dagger = 1 \quad \text{and thus} \quad u^\dagger = u^{-1}. \quad (1.2.14)$$

The direct product $U(k_1) \times U(k_2)$ of ordinary unitary groups is a trivial subgroup of $U(k_1/k_2)$, found by simply putting all anticommuting variables in u to zero. Non-trivial subgroups of the unitary supergroup exist as well. Consider commuting variables, real w_p , $p = 1, \dots, k_1$ and complex z_{pj} , $p = 1, \dots, k_1$, $j = 1, 2$. We introduce the real and quaternion-real supervectors

$$\psi = \begin{bmatrix} w_1 \\ \vdots \\ w_{k_1} \\ \zeta_1 \\ \zeta_1^* \\ \vdots \\ \zeta_{k_2} \\ \zeta_{k_2}^* \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} z_{11} & -z_{12}^* \\ z_{12} & z_{11}^* \\ \vdots & \vdots \\ z_{k_1 1} & -z_{k_1 2}^* \\ z_{k_1 2} & z_{k_1 1}^* \\ \zeta_1^* & -\zeta_1 \\ \vdots & \vdots \\ \zeta_{k_2}^* & -\zeta_{k_2} \end{bmatrix}. \quad (1.2.15)$$

The unitary-ortho-symplectic subgroup of the unitary supergroup leaves the lengths of ψ invariant: $UOSp(k_1/2k_2)$ the length of the first and $UOSp(2k_1/k_2)$ the length of the second supervector in Eq. (1.2.15). Due to the quaternion structure in the commuting entries of the second supervector, the proper scalar product reads $\text{tr } \psi^\dagger \psi$. The trivial ordinary subgroups are $O(k_1) \times USp(2k_2) \subset UOSp(k_1/2k_2)$ and $USp(2k_1) \times O(k_2) \subset UOSp(2k_1/k_2)$, respectively.

As in the ordinary case, non-compact supergroups result from the requirement that the bilinear form $\psi^\dagger L \psi$ remains invariant. The metric L is without loss of generality diagonal and only contains ± 1 . We then have $u^\dagger L u = L$.

A Hermitean supermatrix σ is diagonalized by a supermatrix $u \in U(k_1/k_2)$,

$$\sigma = u^{-1} s u \quad \text{with} \quad s = \text{diag}(s_{11}, \dots, s_{k_1 1}, s_{12}, \dots, s_{k_2 2}) . \quad (1.2.16)$$

All eigenvalues s_{pj} are real commuting. Zirnbauer [Zir96a] gave a classification of the Riemannian symmetric superspaces. The Hermitean symmetric superspace is denoted A|A. Of interest are also the symmetric superspaces AI|AII and AII|AI. The former consists of the $k_1/2k_2 \times k_1/2k_2$ supermatrices $\sigma = u^{-1} s u$ with $u \in \text{UOSp}(k_1/2k_2)$ and with $s = \text{diag}(s_{11}, \dots, s_{k_1 1}, s_{12}, s_{12}, \dots, s_{k_2 2}, s_{k_2 2})$, the latter of the $2k_1/k_2 \times 2k_1/k_2$ supermatrices σ with $u \in \text{UOSp}(2k_1/k_2)$ and with $s = \text{diag}(s_{11}, s_{11}, \dots, s_{k_1 1}, s_{k_1 1}, s_{12}, \dots, s_{k_2 2})$.

1.2.4 Derivatives and Integrals

Since anticommuting variables cannot be represented by numbers, there is nothing like a Riemannian integral over anticommuting variables either. The Berezin integral [Ber87] is formally defined by

$$\int d\zeta_p = 0 \quad \text{and} \quad \int \zeta_p d\zeta_p = \frac{1}{\sqrt{2\pi}} , \quad (1.2.17)$$

and accordingly for the complex conjugates ζ_p^* . The normalization involving $\sqrt{2\pi}$ is a common, but not the only convention used. The differentials $d\zeta_p$ have all the properties of anticommuting variables collected in Sec. 1.2.1. Thus, the Berezin integral of the function (1.2.6) is essentially the highest order coefficient, more precisely $f_{1\dots 11\dots 1}/(2\pi)^k$ apart from an overall sign determined by the chosen order of integration. For example, we have

$$\iint \exp(a\zeta_p^* \zeta_p) d\zeta_p d\zeta_p^* = \frac{a}{2\pi} . \quad (1.2.18)$$

This innocent-looking formula is at the heart of the Supersymmetry method: Anticipating the later discussion, we notice that we would have found the inverse of the right hand side for commuting integration variables z_p instead of ζ_p .

One can also define a derivative as the discrete operation $\partial\zeta_p/\partial\zeta_q = \delta_{pq}$. To avoid ambiguities with signs, one should distinguish left and right derivatives. Obviously, derivative and integral coincide apart from factors. Mathematicians often prefer to think of the Berezin integral as a derivation. In physics, however, the interpretation as integral is highly useful as seen when changing variables. We first consider the k_2 vectors ζ and $\eta = a\zeta$ of anticommuting variables where a is an ordinary complex $k_2 \times k_2$ matrix. From the definition (1.2.17) we conclude $d[\eta] = \det^{-1} a d[\zeta]$. This makes it plausible that the change of

variables in ordinary space generalizes for an arbitrary transformation $\chi = \chi(\psi)$ of supervectors in the following manner: Let y be the vector of commuting and η be the vector of anticommuting variables in χ , then we have

$$d[\chi] = \text{sdet} \frac{\partial \chi}{\partial \psi^T} d[\psi] = \text{sdet} \begin{bmatrix} \partial y / \partial z^T & \partial y / \partial \zeta^T \\ \partial \eta / \partial z^T & \partial \eta / \partial \zeta^T \end{bmatrix} d[\psi] \quad (1.2.19)$$

with $d[\chi] = d[y]d[\eta]$ and $d[\psi] = d[z]d[\zeta]$. The Jacobian in superspace is referred to as Berezinian. Absolute value signs are not needed if we agree to only transform right-handed into right-handed coordinate systems. Changes of variables in superspace can lead to boundary contributions which have no analog in ordinary analysis. In physics, they are referred to as Efetov–Wegner terms, see Ref. [Rot87] for a mathematical discussion.

Importantly, the concept of the δ function has a meaningful generalization in superspace. An anticommuting variable ζ_p acts formally as δ function when integrating it with any function $f(\zeta_p)$, hence $\delta(\zeta_p) = \sqrt{2\pi}\zeta_p$. More complicated are expressions of the form $\delta(y - \zeta^\dagger \zeta)$ with an ordinary commuting variable y and a k component vector of complex anticommuting variables ζ . To make sense out of it, it has to be interpreted as

$$\delta(y - \zeta^\dagger \zeta) = \sum_{\kappa=0}^k \frac{(-1)^\kappa}{\kappa!} \delta^{(\kappa)}(y) (\zeta^\dagger \zeta)^\kappa. \quad (1.2.20)$$

This is a terminating power series, because $(\zeta^\dagger \zeta)^\kappa = 0$ for $\kappa > k$.

1.3 Supersymmetric Representation

Several problems in particle physics would be solved if each Boson had a Fermionic and each Fermion had a Bosonic partner. A review of this Supersymmetry can be found in Ref. [Mar05]. Although mathematically the same, Supersymmetry in condensed matter physics and Random Matrix Theory has a completely different interpretation: the commuting and anticommuting variables do not represent Bosons or Fermions, that is, physical particles. Rather, they are highly convenient bookkeeping devices making it possible to drastically reduce the number of degrees of freedom in the statistical model. Since as many commuting as anticommuting variables are involved, one refers to it as Super–“symmetry” — purely formally just like in particle physics. In 1979, Parisi and Sourlas [Par79] introduced superspace concepts to condensed matter physics. Three years later, Efetov [Efe82] constructed the supersymmetric non-linear σ model for the field theory describing electron transport in disordered systems. Efetov and his coworkers developed many of the tools and contributed a large body of work on Supersymmetry [Efe83]. The first applications of Supersymmetry to random matrices, that is, in the language of condensed matter physics,

to the zero-dimensional limit of a field theory, were given by Verbaarschot and Zirnbauer [Ver85a] and by Verbaarschot, Zirnbauer and Weidenmüller [Ver85b]. Reviews can be found in Refs. [Efe97, Guh98, Mir00], see also the chapters on chiral Random Matrix Theory and on scattering.

1.3.1 Ensemble Average

Using Supersymmetry, the ensemble average in the generating function (1.1.8) is straightforward. We begin with the unitary case $\beta = 2$ and express the determinants as Gaussian integrals

$$\begin{aligned} \frac{(2\pi)^N}{\det(H - x_p^\pm + J_p)} &= \int d[z_p] \exp \left(i L_p z_p^\dagger (H - x_p^\pm + J_p) z_p \right) \\ \frac{\det(H - x_p^\pm + J_p)}{(2\pi)^N} &= \int d[\zeta_p] \exp \left(i \zeta_p^\dagger (H - x_p^\pm - J_p) \zeta_p \right) \end{aligned} \quad (1.3.1)$$

over altogether k vectors z_p , $p = 1, \dots, k$ with N complex commuting entries and k vectors ζ_p , $p = 1, \dots, k$ with N complex anticommuting entries. When integrating over the commuting variables, the imaginary increment is needed for convergence, for the integrals over anticommuting variables, convergence is never a problem. Hence we may write the metric tensor in the form $L = \text{diag}(L_1, \dots, L_k, 1, \dots, 1)$. Collecting all H dependences, the ensemble average in Eq. (1.1.8) amounts to calculating

$$\Phi(K) = \int d[H] P(H) \exp(i \text{tr} H K) . \quad (1.3.2)$$

where the $N \times N$ matrix K assembles dyadic products of the vectors z_p and ζ_p ,

$$K = \sum_{p=1}^k \left(L_p z_p z_p^\dagger - \zeta_p \zeta_p^\dagger \right) . \quad (1.3.3)$$

For all L , this is a Hermitean matrix $K^\dagger = K$.

We now turn to the orthogonal case $\beta = 1$. At first sight it seems irrelevant whether H is Hermitean or real-symmetric in the previous steps. However, the Fourier transform (1.3.2) only affects the real part of K , because the imaginary part of K drops out in $\text{tr} H K$ if H is real-symmetric. Thus, instead of the Gaussian integrals (1.3.1), we rather use

$$\begin{aligned} \frac{\pi^N}{\det(H - x_p^\pm + J_p)} &= \int d[w_p^{(1)}] \exp \left(i L_p w_p^{(1)T} (H - x_p^\pm + J_p) w_p^{(1)} \right) \\ &\quad \int d[w_p^{(2)}] \exp \left(i L_p w_p^{(2)T} (H - x_p^\pm + J_p) w_p^{(2)} \right) \\ \frac{\det(H - x_p^\pm + J_p)}{\pi^N} &= \int d[\zeta_p] \exp \left(i \zeta_p^\dagger (H - x_p^\pm - J_p) \zeta_p \right) \\ &\quad \exp \left(-i \zeta_p^T (H - x_p^\pm - J_p) \zeta_p^* \right) , \end{aligned} \quad (1.3.4)$$

where the to N component vectors $w_p^{(1)}$ and $w_p^{(2)}$ have real entries. For each p , we can construct a $4N$ component supervector out of $w_p^{(1)}$, $w_p^{(2)}$, ζ_p and ζ_p^* whose structure resembles the one of the first of the supervectors (1.2.15), but with a different number of components. Reordering terms, we arrive at the Fourier transform (1.3.2), but now for real-symmetric H and with

$$K = \sum_{p=1}^k \left(L_p w_p^{(1)} w_p^{(1)T} + L_p w_p^{(2)} w_p^{(2)T} - \zeta_p \zeta_p^\dagger + \zeta_p^* \zeta_p^T \right), \quad (1.3.5)$$

which is $N \times N$ real-symmetric as well. For $\beta = 4$, one has to reformulate the steps in such a way that the corresponding K becomes self-dual Hermitean.

1.3.2 Hubbard–Stratonovich Transformation

Due to universality, it suffices to assume a Gaussian probability density function $P(H) \sim \exp(-\beta \text{tr } H^2/2)$ in almost all applications in condensed matter and many-body physics as well as in quantum chaos. Hence the random matrices are drawn from the Gaussian orthogonal (GOE), unitary (GUE) or symplectic ensemble (GSE). The Fourier transform (1.3.2) is then elementary and yields a Gaussian. The crucial property

$$\Phi(K) = \exp \left(-\frac{1}{2\beta} \text{tr } K^2 \right) = \exp \left(-\frac{1}{2\beta} \text{str } B^2 \right) \quad (1.3.6)$$

holds, where B is supermatrix containing all scalar products of the vectors to be integrated over. The second equality sign has a purely algebraic origin. For $\beta = 2$, B has dimension $k/k \times k/k$ and reads

$$B = L^{1/2} \begin{bmatrix} z_1^\dagger z_1 & \cdots & z_1^\dagger z_k & z_1^\dagger \zeta_1 & \cdots & z_1^\dagger \zeta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ z_k^\dagger z_1 & \cdots & z_k^\dagger z_k & z_k^\dagger \zeta_1 & \cdots & z_k^\dagger \zeta_k \\ -\zeta_1^\dagger z_1 & \cdots & -\zeta_1^\dagger z_k & -\zeta_1^\dagger \zeta_1 & \cdots & -\zeta_1^\dagger \zeta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ -\zeta_k^\dagger z_1 & \cdots & -\zeta_k^\dagger z_k & -\zeta_k^\dagger \zeta_1 & \cdots & -\zeta_k^\dagger \zeta_k \end{bmatrix} L^{1/2}. \quad (1.3.7)$$

While K is Hermitean, the square roots $L^{1/2}$ destroy this property for B , since $L^{1/2}$ can have imaginary units i as entries, B is Hermitean only for $L = 1$. In general, B is in a deformed (non-compact) form of the symmetric superspace $A|A$. For $\beta = 1, 4$, the supermatrix B has dimension $2k/2k \times 2k/2k$ and it is in deformed (non-compact) forms of the symmetric superspaces $AI|AII$ and $AII|AI$, respectively. We give the explicit forms later on. The identity (1.3.6) states the keystone of the Supersymmetry method. The original model in the

space of ordinary $N \times N$ matrices is mapped onto a model in space of supermatrices whose dimension is proportional to k , which is the number of arguments in the k -point correlation function.

The Gaussians (1.3.6) contain the vectors, that is, their building blocks to fourth order. To make analytical progress, a Hubbard–Stratonovich transformation in superspace is used,

$$\exp\left(-\frac{1}{2\beta}\text{str } B^2\right) = c^{(\beta)} \int \exp\left(-\frac{\beta}{2}\text{str } (L\sigma)^2\right) \exp\left(i\text{str } L^{1/2}\sigma L^{1/2}B\right) d[\sigma] , \quad (1.3.8)$$

where $c^{(2)} = 2^{k(k-1)}$ and $c^{(\beta)} = 2^{k(4k-3)/2}$ for $\beta = 1, 4$. We notice the appearance of the matrices L and $L^{1/2}$ in (1.3.8). For $L = 1$, the supermatrices σ and B have the same symmetries. However, as already observed in the early eighties for models in ordinary space [Sch80, Pru82], this choice is impossible for $L \neq 1$, because it would render the integrals divergent. There are two ways out of this problem. One either constructs a proper explicit parameterization of σ or one inserts the matrices L and $L^{1/2}$ according to (1.3.8). A mathematically satisfactory understanding of these issues was put forward only recently in Ref. [Fyo08].

Another important remark is called for. Because of the minus sign in the supertrace (1.2.12), a Wick rotation is needed to make the integral convergent. It formally amounts to replacing the lower right block of σ , that is, b in Eq. (1.2.8), with ib . Apart from that, the metric L is also needed for convergence reason. Now the vectors appear in second order. They can be ordered in one large supervector Ψ . For $\beta = 2$ it has the form (1.2.9) with $k_1 = k_2 = kN$, for $\beta = 1$ it has the first of the forms (1.2.15) with $k_1 = 2kN$, $k_2 = kN$ and for $\beta = 4$ it has the second of the forms (1.2.15) with $k_1 = kN$, $k_2 = 2kN$. The integral to be done is then seen to be the Gaussian integral in superspace

$$\begin{aligned} & \int \exp\left(i\Psi^\dagger \left(L^{1/2}(L^{1/2}\sigma L^{1/2} - x^\pm - J)L^{1/2} \otimes 1_N\right) \Psi\right) d[\Psi] \\ & = \text{sdet }^{-N\beta/2\gamma}(\sigma L - x^\pm - J) , \end{aligned} \quad (1.3.9)$$

where the power N is due to the direct product structure. We eventually find

$$Z_k(x + J) = c^{(\beta)} \int \exp\left(-\frac{\beta}{2}\text{str } (L\sigma)^2\right) \text{sdet }^{-N\beta/2\gamma}(\sigma L - x^\pm - J) d[\sigma] \quad (1.3.10)$$

as supersymmetric representation of the generating function. The average over the $N \times N$ ordinary matrix H has been traded for an average over the matrix σ whose dimension is proportional to k , that is, independent of N .

1.3.3 Matrix δ Functions and an Alternative Representation

In Refs. [Leh95, Hac95], a route alternative to the one outlined in Sec. 1.3.2 was taken. These authors used matrix δ functions in superspace and their Fourier

representation to express functions $f(B)$ of the supermatrix B in the form

$$f(B) = \int f(\rho) \delta(\rho - B) d[\rho] = c^{(\beta)^2} \int d[\rho] f(\rho) \int d[\sigma] \exp(-i \text{str } \sigma(\rho - B)) , \quad (1.3.11)$$

where auxiliary integrals over supermatrices ρ and σ are introduced. For simplicity, we only consider $L = 1$ here. The function $\delta(\rho - B)$ is the product of the δ functions of all independent variables. As discussed in Sec. 1.2.4, it is well-defined. For all functions f , formula (1.3.11) renders the integration over the supervector Ψ Gaussian. When studying Gaussian averages of ratios of characteristic polynomials, Fyodorov [Fyo02] built upon such insights to construct an alternative representation for the generating function. He employs a standard Hubbard–Stratonovich transformation for the lower right block of the supermatrix B in Eq. (1.3.7) which contains the scalar products $\zeta_p^\dagger \zeta_q$. He then inserts a δ function in the space of ordinary matrices to carry out the integrals over the vectors z_p . Although Supersymmetry is used, the generating function is finally written as an integral over two ordinary matrices with commuting entries. In this derivation, the Ingham–Siegel integral

$$I^{(\text{ord})}(R) = \int_{S>0} \exp(-\text{tr } RS) \det^m S d[S] \sim \frac{1}{\det^{m+N} R} \quad (1.3.12)$$

for ordinary Hermitean $N \times N$ matrices R and S appears, where $m \geq 0$.

1.3.4 Generalized Hubbard–Stratonovich Transformation and Superbosonization

Is Supersymmetry only applicable to Gaussian probability density functions $P(H)$? — In Ref. [Hac95], Supersymmetry and asymptotic expansions were used to prove universality for arbitrary $P(H)$. The concept of superbosonization was put forward in Ref. [Efe04] and applied in Ref. [Bun07] to a generalized Gaussian model comprising a variety of correlations between the matrix elements. Extending the concept of superbosonization, a full answer to the question posed above was given in two different but related approaches in Refs. [Guh06, Kie09a] and [Lit08]: An exact supersymmetric representation exists for arbitrary, well-behaved $P(H)$. As the equivalence of the two approaches was proven in Ref. [Kie09b], we follow the line of arguing in Refs. [Guh06, Kie09a]. For $\beta = 2$, we define the $N \times 2k$ rectangular supermatrix

$$A = [z_1 \cdots z_k \ \zeta_1 \cdots \zeta_k] , \quad (1.3.13)$$

where the z_p , $p = 1, \dots, k$ and ζ_p , $p = 1, \dots, k$ are N component vectors with complex commuting and anticommuting entries, respectively. We also define the $N \times 4k$ supermatrix

$$A = [z_1 \ z_1^* \cdots z_k \ z_k^* \ \zeta_1 \ \zeta_1^* \cdots \zeta_k \ \zeta_k^*] \quad (1.3.14)$$

for $\beta = 1$ and eventually the $2N \times 4k$ supermatrix

$$A = \begin{bmatrix} z_1 & -z_1^* & \dots & z_k & -z_k^* & \zeta_1 & -\zeta_1^* & \dots & \zeta_k & -\zeta_k^* \\ z_1 & z_1^* & & z_k & z_k^* & \zeta_1 & \zeta_1^* & & \zeta_k & \zeta_k^* \end{bmatrix} \quad (1.3.15)$$

for $\beta = 4$. This enables us to write the ordinary matrix K introduced in Sec. 1.3.1 and the supermatrix B introduced in Sec. 1.3.2 for all β in the form

$$\begin{aligned} K &= ALA^\dagger = (AL^{1/2})(L^{1/2}A^\dagger) \\ B &= (L^{1/2}A^\dagger)(AL^{1/2}) = L^{1/2}A^\dagger AL^{1/2}. \end{aligned} \quad (1.3.16)$$

For $\beta = 2$, we recover Eq. (1.3.7). This algebraic duality between ordinary and superspace has far-reaching consequences. One realizes [Guh91, Guh06, Lit08] that the integral (1.3.2) is the Fourier transform in matrix space of every, arbitrary probability density function $P(H)$ and that $\Phi(K)$ is the corresponding characteristic function. Since we assume that $P(H)$ is rotation invariant, the same must hold for $\Phi(K)$. Hence, $\Phi(K)$ only depends on the invariants $\text{tr } K^m$, $m = 1, 2, 3, \dots$. Due to cyclic invariance of the trace, the duality (1.3.16) implies for all m the crucial identity

$$\text{tr } K^m = \text{str } B^m, \quad \text{such that} \quad \Phi(K) = \Phi(B). \quad (1.3.17)$$

Hence, viewed as a function of the matrix invariants, Φ is a function in ordinary and in superspace. We now employ formula (1.3.11) for $\Phi(K) = \Phi(B)$, do the Gaussian Ψ integrals as usual find for the generating function

$$Z_k(x + J) = c^{(\beta)^2} \int \exp(-i \text{tr}(x + J)L\rho) \Phi(\rho) I(\rho) d[\rho] \quad (1.3.18)$$

with $I(\rho)$ being a supersymmetric version of the Ingham–Siegel integral. The supermatrices ρ and σ have the same sizes and symmetries as B . A convolution theorem in superspace yields the second form

$$Z_k(x + J) = \int \Pi(\sigma) \text{sdet}^{-N\beta/2\gamma} (\sigma L - x^\pm - J) d[\sigma], \quad (1.3.19)$$

where $\Pi(\sigma)$ is the superspace Fourier backtransform of the characteristic function $\Phi(\rho)$. It plays the rôle of the probability density function for the supersymmetric representation. To apply these general results for exact calculations, explicit knowledge of either $\Phi(\rho)$ or $\Pi(\sigma)$ is necessary.

1.3.5 More Complicated Models

Most advantageously, Supersymmetry allows one to make progress in important and technically challenging problems beyond the invariant and factorizing ensembles, for example:

- Invariant, but non-factorizing ensembles. The probability density function $P(H)$ has the property (1.1.1), but not the property (1.1.4). They can, in principle, be treated with the results of Sec. 1.3.4.
- Sparse or banded random matrices [Fyo91, Mir91], see chapter 23. The probability density function $P(H)$ lacks the invariance property (1.1.1).
- Crossover transitions or external field models. One is interested in the eigenvalue correlations of the matrix $H(\alpha) = H^{(0)} + \alpha H$, where H is a random matrix as before and where $H^{(0)}$ is either a random matrix with symmetries different from H or a fixed matrix. The parameter α measures the relative strength. As the resolvent in question is now $(x_p^- - H(\alpha))^{-1} = (x_p^- - (H^{(0)} + \alpha H))^{-1}$, we have to replace H by $H(\alpha)$ in the determinants in Eq. (1.1.8), but not in the probability density function $P(H)$ which usually is chosen invariant, see Ref. [Guh96b].
- Scattering theory and other problems, where matrix elements of the resolvents enter [Ver85b], see chapter 2 on history and chapter 34 on scattering. In the Heidelberg formalism [Mah69], scattering is modeled by coupling an effective Hamiltonian which describes the interaction zone to the scattering channels. The resolvent is then $(x_p^- - H + iW)^{-1}$ where the $N \times N$ matrix W contains information about the channels. One has to calculate averages of products of matrix elements $[(x_p^- - H + iW)^{-1}]_{nm}$. To make that feasible, the source variables J_p have to be replaced by $N \times N$ source matrices \tilde{J}_p and instead of the derivatives (1.1.7), one must calculate derivatives with respect to the matrix elements $\tilde{J}_{p,nm}$. The probability density function $P(H)$ is unchanged.
- Field theories for disordered systems, see Refs. [Efe83, Efe97].

Of course, these and other non-invariant problems can not only be studied with the Supersymmetry method, other techniques ranging from perturbative expansions, asymptotic analysis to orthogonal polynomials supplemented with group integrals are applied as well, see chapters 4., 5. and 6. Nevertheless, the drastic reduction in the numbers of degrees of freedom, which is the key feature of Supersymmetry, often yields precious structural insights into the problem.

1.4 Evaluation and Structural Insights

To evaluate the supersymmetric representation, a large N expansion, the celebrated non-linear σ model, is used in the vast majority of applications. We also sketch a method of exact evaluation which amounts to a diffusion process in superspace. Throughout, we focus on the structural aspects. A survey of the

numerous results for specific systems is beyond the scope of this contribution, we refer the reader to the reviews in Refs. [Efe97, Guh98, Mir00].

1.4.1 Non-linear σ Model

The reduction in the numbers of degrees of freedom is borne out in the fact that the dimension N of the original random matrix H is an explicit parameter in Eqs. (1.3.10) and (1.3.19). Hence we can obtain an asymptotic expansion in $1/N$ by means of a saddle point approximation [Efe83, Efe97, Ver85a, Ver85b]. This suffices because one usually is interested in the correlations on the local scale of the mean level spacing. Hence, the saddle point approximation goes hand in hand with the unfolding. The result of this procedure is the supersymmetric non-linear σ model. We consider the two-point function $k = 2$. The integrand in Eq. (1.3.10) is written as $\exp(-F(x + J))$ with the free energy

$$F(x + J) = \frac{\beta}{2} \text{str}(L\sigma)^2 + \frac{N\beta}{2\gamma} \text{str} \ln(\sigma L - x^\pm - J) , \quad (1.4.1)$$

which is also referred to as Lagrangean. We introduce center $\bar{x} = (x_1 + x_2)/2$ and difference $\Delta x = x_2 - x_1$ of the arguments. In the large N limit, $\xi = \Delta x/D$ has to be held fixed where $D \sim 1/\sqrt{N}$ is the local mean level spacing. Hence, when determining the saddle points, we may set $\Delta x = 0$ such that $x = \bar{x}1$. Moreover, as we may choose the source variables arbitrarily small, we set $J = 0$ as well. Since all symmetry breaking terms are gone, the free energy $F(x)$ with $x = \bar{x}1$ is invariant under rotations of σ which obey the metric L . Thus, variation of $F(x)$ with respect to σ yields the scalar equation

$$s_0(\bar{x} - s_0) = \frac{N}{2\gamma} , \quad \text{such that} \quad s_0 = \frac{1}{2} \left(\bar{x} \pm i \sqrt{\frac{2N}{\gamma} - \bar{x}^2} \right) \quad (1.4.2)$$

inside the spectrum, $|\bar{x}| \leq \sqrt{2N/\gamma}$. This is the famous Pastur equation and its solution s_0 [Pas72]. The latter is proportional to the large N one-point function whose imaginary part is the Wigner semicircle. We arrive at the important insight that the one-point function provides the stable points of the supersymmetric representation, the correlations on the local scale are the fluctuations around it. To make this more precise, we recall the result of Schäfer and Wegner [Sch80] for the non-linear σ model in ordinary space: when doing the large N limit as sketched above, the imaginary increments of the arguments x_1 and x_2 must lie on different sides of the real axis. Otherwise, the connected part of the two-point function cannot be obtained as seen from a contour-integral argument. Hence, the metric L must not be proportional to the unit matrix, the groups involved are non-compact and a hyperbolic symmetry is present. This carries over to the commuting degrees of freedom in superspace [Efe82],

the groups are $U(1, 1/2)$ for $\beta = 2$ and $UOSp(2, 2/4)$ for $\beta = 1, 4$. The full saddle point manifold is found to be given by all non-compact rotations of $\sigma_0 = \bar{x}/2 + i\sqrt{2N/\gamma - \bar{x}^2}L/2$ which leave $F(\bar{x}1)$ invariant. One parameterizes the group as $u = u_0v$ with u_0 in the direct product $U(1/1) \times U(1/1)$ for $\beta = 1$ and in $UOSp(2/2) \times UOSp(2/2)$ for $\beta = 1, 4$ and with v in the coset

$$\frac{U(1, 1/2)}{U(1/1) \times U(1/1)} \quad \text{for} \quad \beta = 2 ,$$

$$\frac{UOSp(2, 2/4)}{UOSp(2/2) \times UOSp(2/2)} \quad \text{for} \quad \beta = 1, 4 . \quad (1.4.3)$$

As u_0 and L commute, the saddle point manifold is $v^{-1}\sigma_0v$, that is, essentially $Q = v^{-1}Lv$ with the crucial property $Q^2 = 1$. To calculate the correlations, we must re-insert the symmetry breaking terms Δx and J into the free energy. We put $\sigma = v^{-1}\sigma_0v + \delta\sigma$ and expand to second order in the variables $\delta\sigma$ which are referred to as massive modes. They are integrated out in the generating function (1.3.10) as Gaussian integrals. One is left with integrals over the coset manifold, that is, over the Goldstone modes. On the unfolded scale, the two-point correlation functions (1.1.3) acquire the form $1 - Y_2(\xi)$. The two-level cluster functions read

$$Y_2(\xi) = -\text{Re} \int \exp(i\xi \text{str} QL) \text{str} M_1 QL \text{str} M_2 QL d\mu(Q) , \quad (1.4.4)$$

where $d\mu(Q)$ is the invariant measure on the saddle point manifold. The matrices M_i , $i = 1, 2$ result from the derivatives with respect to the source variables, M_i is found by formally setting $J_i = 1$ and $J_l = 0$, $l \neq i$, in the matrix J . The expressions (1.4.4) can be reduced to two radial integrals on the coset manifold for the GUE and to three such integrals for GOE and GSE. Efetov [Efe83] discovered Eq. (1.4.4) when taking the zero-dimensional limit of his supersymmetric non-linear σ model for electron transport in disordered mesoscopic systems. He thereby established a most fruitful link between Random Matrix Theory and mesoscopic physics.

The non-linear σ model, particularly its mathematical aspects, was recently reviewed in Ref. [Zir06].

1.4.2 Eigenvalues and Diffusion in Superspace

The supersymmetric representation and Fyodorov's alternative representation of Sec. 1.3 are exact for finite N . In some situations, it is indeed possible and advantageous to evaluate them without using the non-linear σ model. It has been shown for the supersymmetric representation [Guh06, Kie09a] that the imaginary increments of the arguments may then all lie on the same side of the real axis. We have $L = 1$ and all groups are compact. As we aim

at structural aspects, we consider the crossover transitions involving $H(\alpha) = H^{(0)} + \alpha H$ as discussed in Sec. 1.3.5. We introduce the fictitious time $t = \alpha^2/2$. Dyson [Dys62, Dys72] showed that the eigenvalues of $H(t)$ follow a Brownian motion in t which implies that their probability density function is propagated by a diffusion equation. Without any loss of information, Supersymmetry reduces this stochastic process to a Brownian motion in a much smaller space [Guh96b]. It is precisely the radial part of the Riemannian symmetric superspaces discussed in Sec. 1.2.3. The quantity propagated is then the generating function $Z_k(x + J, t)$ of the correlations. The initial condition

$$Z_k^{(0)}(s) = \int P^{(0)}(H^{(0)}) \text{sdet}^{-1}(s \otimes 1 + 1 \otimes H^{(0)}) d[H^{(0)}] \quad (1.4.5)$$

is arbitrary, it includes ensembles, but also a fixed matrix $H^{(0)}$ if the probability density $P^{(0)}(H^{(0)})$ is chosen accordingly. Due to the direct product structure, $Z_k^{(0)}(s)$ is rotation invariant. The diagonal matrix s is in the above mentioned radial space, such that $\sigma = u^{-1}su$, see Sec. 1.2.3. For $\beta = 2$, this space coincides with $x + J$, for $\beta = 1, 4$, it is slightly larger. The generating function is then a convolution in the radial space,

$$Z_k(r, t) = \int \Gamma_k(s, r, t) Z_k^{(0)}(s) B_k(s) d[s] . \quad (1.4.6)$$

When going to eigenvalue–angle coordinates $\sigma = u^{-1}su$ Berezinians $B_k(s)$ occur analogous to $|\Delta_N(X)|^\beta$ in ordinary space. The propagator is the supergroup integral

$$\Gamma_k(s, r, t) = c^{(\beta)} \exp\left(-\frac{\beta}{4t} \text{str}(s^2 + r^2)\right) \int \exp\left(-\frac{\beta}{2t} \text{str} u^{-1}sur\right) d\mu(u) . \quad (1.4.7)$$

For $\beta = 2$ and all k , this integral is known explicitly [Guh91, Guh96a]. Unfortunately, for $\beta = 1, 4$ the available result [Guh02] is handy only for $k = 1$ but cumbersome for $k = 2$.

It is a remarkable inherent feature of Supersymmetry that the propagator and thus the diffusion process of $Z_k(r, t)$ on the original scale in t carries over unchanged to the unfolded scale when introducing the proper time $\tau = t/D^2$. The initial condition is the unfolded large N limit of $Z_k^{(0)}(s)$. Moreover and in contrast to the hierarchical equations for the correlation functions [Fre88], the Brownian motion in superspace for the generating functions is diagonal in k .

1.5 Circular Ensembles and Color–Flavor Transformation

In many physics applications, the random matrices H model Hamiltonians, implying that they are either real symmetric, Hermitean or quaternion real.

Mathematically speaking, they are in the non-compact forms of the corresponding symmetric spaces. However, if one aims at modeling scattering, it is often useful to work with random unitary matrices S , taken from the compact forms of these symmetric spaces [Dys62]. These are $U(N)/O(N)$, $U(N)$ and $U(2N)/Sp(2N)$, respectively, leading to the circular orthogonal, unitary and symplectic ensemble COE, CUE and CSE which are labeled $\beta = 1, 2, 4$. The phase angles of S play the same rôle as the eigenvalues of H . Due to the compactness, no Gaussian or other confining function is needed and the probability density function is just the invariant measure on the symmetric space in question. On the local scale of the mean level spacing, the correlations coincide with those of the Gaussian ensembles [Dys62, Meh04].

Zirnbauer [Zir96b] showed how to apply Supersymmetry to the circular ensembles. His approach works for all three symmetry classes, but for simplicity we only discuss the CUE which consists of the unitary matrices $S = U \in U(N)$. Consider the generating function

$$Z_{k_+k_-}(\vartheta, \varphi) = \int d\mu(U) \prod_{p=1}^{k_+} \frac{\det(1 - \exp(i\varphi_{+p})U)}{\det(1 - \exp(i\vartheta_{+p})U)} \prod_{q=1}^{k_-} \frac{\det(1 - \exp(i\varphi_{-q})U^\dagger)}{\det(1 - \exp(i\vartheta_{-q})U^\dagger)}, \quad (1.5.1)$$

where $d\mu(U)$ is the invariant measure on $U(N)$. To derive the correlation functions $R_k(\vartheta_1, \dots, \vartheta_k)$ one sets $k_+ = k_- = k$, takes derivatives with respect to the variables $\varphi_{\pm p}$ and puts certain combinations of variables $\varphi_{\pm p}$ and $\vartheta_{\pm p}$ equal. The variables $\vartheta_{\pm p}$ in Eq. (1.5.1) have small imaginary increments to prevent $Z_{k_+k_-}(\vartheta, \phi)$ from becoming singular.

Since the Hubbard–Stratonovich transformation of Sec. 1.3.2 cannot be employed to construct a supersymmetric representation of $Z_{k_+k_-}(\vartheta, \phi)$, Zirnbauer [Zir96b] developed the color–flavor transformation based on the identity

$$\int d\mu(U) \exp(\Psi_{+pj}^{n*} U^{nm} \Psi_{+pj}^m + \Psi_{+qj}^{n*} U^{nm*} \Psi_{+qj}^m) = \int d[\Lambda] d[\tilde{\Lambda}] \text{sdet}^N (1 - \tilde{\Lambda} \Lambda) \exp(\Psi_{+pj}^{n*} \Lambda_{pjql} \Psi_{-ql}^n + \Psi_{-ql}^{m*} \tilde{\Lambda}_{qlpj} \Psi_{+pj}^m). \quad (1.5.2)$$

which transforms an integral over the ordinary group $U(N)$ into an integral over $k_+/k_+ \times k_-/k_-$ rectangular supermatrices Λ and $\tilde{\Lambda}$ parameterizing the coset space $U(k_+ + k_-/k_+ + k_-)/U(k_+/k_+) \times U(k_-/k_-)$. The integrals depend on the supertensor Ψ with components Ψ_{+pj}^n . The indices $n, m = 1, \dots, N$ label the elements of U in ordinary space. The indices pj and ql are superspace indices with $p = 1, \dots, k_+$, $q = 1, \dots, k_-$ and with $j, l = 1, 2$ labeling the four blocks of the supermatrices, see Eq. (1.2.8). Summation convention applies. The superfields Ψ are used to express the determinants in Eq. (1.5.1) as Gaussian integrals. After the color–flavor transformation, they are integrated out again.

As the name indicates, the color–flavor transformation is naturally suited for applications in lattice gauge theories where U is in the color gauge group, see Ref. [Nag01]. In Ref. [Wei05], the color–flavor transformation was derived for the gauge group $SU(N)$ relevant in lattice quantum chromodynamics.

1.6 Concluding Remarks

Apart from the wealth of results for specific physics systems which had to be left out, some important conceptual issues could not be discussed here either due to lack of space: we only mentioned applications of the other Riemannian symmetric superspaces [Zir96a] to Andreev scattering and chiral Random Matrix Theory. As random matrix approaches are now ubiquitous in physics and beyond, one may also expect that the Supersymmetry method spreads out accordingly. From a mathematical viewpoint, various aspects deserve further clarifying studies, in the present context most noticeably the theory of supergroups and harmonic analysis on superspaces.

ACKNOWLEDGEMENTS:

I thank Mario Kieburg and Heiner Kohler for helpful discussions. I acknowledge support from Deutsche Forschungsgemeinschaft within Sonderforschungsbereich Transregio 12 “Symmetries and Universality in Mesoscopic Systems”.

References

- [Ber87] F.A. Berezin, *Introduction to Supermanifolds*, Reidel, Dordrecht 1987
- [Bun07] J.E. Bunder, K.B. Efetov, V.E. Kravtsov, O.M. Yevtushenko and M.R. Zirnbauer, *J. Stat. Phys.* **129** (2007) 809
- [Dys62] F.J. Dyson, *J. Math. Phys.* **3** (1962) 140; 157; 166; 1199
- [Dys72] F.J. Dyson, *J. Math. Phys.* **13** (1972) 90
- [Efe82] K.B. Efetov, *Zh. Eksp. Teor. Fiz.* **82** (1982) 872 [*Sov. Phys. JETP* **55** (1982) 514]
- [Efe83] K.B. Efetov, *Adv. in Phys.* **32** (1983) 53
- [Efe97] K. B. Efetov, *Supersymmetry in Disorder and Chaos*, Cambridge University Press, Cambridge 1997
- [Efe04] K.B. Efetov, G. Schwiete and K. Takahashi, *Phys. Rev. Lett.* **92** (2004) 026807

- [Fre88] J.B. French, V.K.B. Kota, A. Pandey and S. Tomsovic, Ann. Phys. (NY) **181** (1988) 198
- [Fyo91] Y.V. Fyodorov and A.D. Mirlin, Phys. Rev. Lett. **67** (1991) 2405
- [Fyo02] Y.V. Fyodorov, Nucl. Phys. **B621** (2002) 643
- [Fyo08] Y.V. Fyodorov, Y. Wei and M.R. Zirnbauer, J. Math. Phys. **9** (2008) 053507
- [Guh91] T. Guhr, J. Math. Phys. **32** (1991) 336
- [Guh96a] T. Guhr, Commun. Math. Phys. **176** (1996) 555
- [Guh96b] T. Guhr, Ann. Phys. (NY) **250** (1996) 145
- [Guh98] T. Guhr, A. Müller-Groeling and H.A. Weidenmüller, Phys. Rep. **299** (1998) 189
- [Guh02] T. Guhr and H. Kohler, J. Math. Phys. **43** (2002) 2741
- [Guh06] T. Guhr, J. Phys. **A39** (2006) 13191
- [Hac95] G. Hackenbroich and H.A. Weidenmüller, Phys. Rev. Lett. **74** (1995) 4418
- [Kac77] V.G. Kac, Commun. Math. Phys. **53** (1977) 31
- [Kie09a] M. Kieburg, J. Grönqvist and T. Guhr, J. Phys. **A42** (2009) 275205
- [Kie09b] M. Kieburg, H.J. Sommers and T. Guhr, J. Phys. **A42** (2009) 275206
- [Leh95] N. Lehmann, D. Saher, V.V. Sokolov and H.J. Sommers, Nucl. Phys. **A582** (1995) 223
- [Lit08] P. Littlemann, H.J. Sommers and M.R. Zirnbauer, Commun. Math. Phys. **283** (2008) 343
- [Mah69] C. Mahaux and H.A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions*, North-Holland, Amsterdam 1969
- [Mar59] I.L. Martin, Proc. Roy. Soc. **A251** (1959) 536
- [Mar05] S.P. Martin, arXiv:hep-ph/9709356, version 5, 2005
- [Meh04] M.L. Mehta, *Random Matrices*, Academic Press, 3rd Edition, London 2004
- [Mir00] A. Mirlin, Phys. Rep. **326** (2000) 259

- [Mir91] A.D. Mirlin and Y. Fyodorov, J. Phys. **A24** (1991) 2273
- [Nag01] T. Nagao and S.M. Nishigaki, Phys. Rev. **D64** (2001) 014507
- [Par79] G. Parisi and N. Surlas, Phys. Rev. Lett. **43** (1979) 744
- [Pas72] L. Pastur, Theor. Mat. Phys. **10** (1972) 67
- [Pru82] A.M.M. Pruisken and L. Schäfer, Nucl. Phys. **B200** (1982) 20
- [Rot87] M.J. Rothstein, Trans. Am. Math. Soc. **299** (1987) 387
- [Sch80] L. Schäfer and F. Wegner, Z. Phys. **B38** (1980) 113
- [Ver85a] J.J.M. Verbaarschot and M.R. Zirnbauer, J. Phys. **A17** (1985) 1093
- [Ver85b] J.J.M. Verbaarschot, H.A. Weidenmüller and M.R. Zirnbauer, Phys. Rep. **129** (1985) 367
- [Wei05] Y. Wei and T. Wettig, J. Math. Phys. **46** (2005) 072306
- [Zir96a] M.R. Zirnbauer, J. Math. Phys. **37** (1996) 4986
- [Zir96b] M.R. Zirnbauer, J. Phys. **A29** (1996) 7113
- [Zir06] M.R. Zirnbauer, in: *Encyclopedia of Mathematical. Physics*, vol. 5, pp. 151, Elsevier, Amsterdam 2006